

82-104/114 Math Review

This is a review of the kind of math you will be expected to do in this class. It is at approximately the *high-school* sophomore/junior math level. They are the same methods that you employ to balance your checkbook, do your taxes, or figure out how long it takes to drive to Toledo, so non-science majors should be able to handle it.

The purpose of this review is to *refresh* yourself on the basic mathematical tools you will need to work the problems both on tests and in the labs. We realize that some of you might be a little out of practice when it comes to doing math. If this is the case, don't just give up, but come and see us for help. That is what we are here for.

1. Algebra

Algebra is a way of doing math with letters instead of just numbers. The letters are called *variables* if their values change (vary), and *constants* if the values are always the same (constant). So, instead of writing:

$$6 = 2 \times 3$$

we can say:

$$y = ax$$

(it reads “y equals a times x”) if we also say $y = 6$, $a = 2$, and $x = 3$. The two equations are effectively telling us the *same* thing.

Numbers and letters can also be mixed in the same equation. If we say:

$$y = 5x$$

we say the same “ $y = ax$ ” as above, where “a” has been set to a constant value of 5.

We can solve an algebraic equation for *any one* variable, *provided* we know all the others. If $y = 5x$ as above, we can say $y = 10$ if and only if $x = 2$.

We can also arrange any equation in order to solve for any of the variables. The trick is to isolate the variable of interest (the one you are trying to solve for) to one side of an equals sign, with all the other variables and constants on the other. You do this by adding, subtracting, multiplying, or dividing variables and constants from *both* sides of the equation. To solve the above equation $y = ax$ for x :

$$y = ax$$

$$\frac{y}{a} = \frac{ax}{a} \quad \text{Divide both sides by “a”}$$

$$\frac{y}{a} = x \quad \text{The “a” divided by “a” on the right side cancels.}$$

Exponentiation

The most common example of this is “squaring” as in:

$$x^2 = x \times x$$

but there are also others like:

$$x^3 = x \times x \times x$$

$$x^6 = x \times x \times x \times x \times x \times x$$

Since in normal algebra this would be:

$$x^3 = xxx \quad \text{and} \quad x^6 = xxxxxx$$

you can see why exponents were invented! The largest exponent you will ever see in this class is 4.

Roots

Roots are the opposite of exponents. So:

$$\sqrt{x^2} = \sqrt{xx} = x, \quad \text{and} \quad \sqrt[3]{x^3} = \sqrt[3]{xxx} = x$$

Note that the leading 2 in the case of the “square root” (above left) is implied and not written. The rightmost example is called a “cube root.”

Another (more complicated) example is:

$$\sqrt{x^6} = \sqrt{x^3 x^3} = x^3, \quad \text{and} \quad \sqrt[3]{x^{12}} = \sqrt[3]{x^4 x^4 x^4} = x^4$$

The last example works because exponents and roots have the following properties:

- 1) to multiply exponential numbers (of the same variable or constant) you *add* the exponents.
- 2) to divide exponential numbers (of the same variable or constant) you *subtract* the exponents.

Because of this, we can write:

$$(x^3)(x^3) = x^6$$

$$(x^2)(x^2)(x^2) = x^6$$

$$(x^4)(x^{-3}) = x^1 = x$$

$$\frac{x^7}{x^2} = x^{7-2} = x^5$$

Area and Volume

Just a reminder of some formulae:

Area of a:	square	= s^2	s = length of one side
	circle	= r^2	r = radius (1/2 diameter)
	sphere (surface)	= $4 r^2$	r = radius

Volume of a:	cube	= s^3
	sphere	= $\frac{4}{3} r^3$

Note that area is a two dimensional concept, this it involves **squaring**. Volume is a three dimensional concept, so it involves **cubing**.

The Greek letter π (pi) is the ratio of a circle's circumference to its diameter and is a universal constant equal to 3.14159... (3.14 is usually enough for us). It was first derived by Euclid around 4000 years ago and has recently been determined to its Billionth decimal place!

2. Some Words About Units

Units are very important, not only in science, but in everyday "real life" as well. If you are running low on gas somewhere out in the Arizona desert, the ambiguous sign "Next gas 100" can be very important. But is that 100 yards or miles? Getting over a cold would be frustrated by a bottle with the helpful guidelines "Take two twice per day." Two what? Tablespoons? Teaspoons? Bottles? You can only imagine how easy cooking would be without units!

Units can also be very helpful in determining how to get an answer. For example, we know that velocity (or *speed*) is typically measured in miles per hour, so we can guess that the formula for calculating velocity is distance (in miles) divided by (hence the "per") time (in hours).

The three basic properties of an object are *distance* (such as height, depth, or width), *time* (duration), and *mass* (which at the Earth's surface is roughly numerically equivalent to *weight*). In the "English System" (which is only used in the US - figure that one out) these quantities are measured in the following units:

<i>distance:</i>	inches, feet, yards, miles
<i>time:</i>	seconds, minutes, hours, days, years, etc.
<i>mass:</i>	ounces, pounds, tons

All of the other units (like area, velocity, etc.) are derived from these basic units. How do we go back and forth between these units? Well, we know there are 12 inches in a foot. This is called a *conversion factor*:

$$1 \text{ foot} = 12 \text{ inches}$$

So, if we wanted to compute the number of inches in 3 feet we would say:

$$3 \text{ foot(s)} \times \frac{12 \text{ inches}}{1 \text{ foot}} = 36 \text{ inches}$$

Notice two things about the above equation. 1) 1 foot divided by 12 inches equals one, so by multiplying by the conversion factor, we are not really changing anything but the units. 2) the foot units on the left hand side *cancel* to leave us with the units we need, inches. You should always do unit conversions like this; it's a pain to write all those units out just to cross them off, but it will save you time and trouble in the future.

Here are some more conversion factors (you do not need to remember them):

3 ft.	=	1 yd.
24 hrs.	=	1 day
60 sec.	=	1 min.
2000 lbs.	=	1 ton
1 year	=	365.25 days

Notice how some of these conversion factors are really bizarre? Can you recall (without looking) the number of yards in a mile? The number of ounces in a pound? Part of this is because many units (especially distance) were once based on whoever happened to be ruling the area at that time (e.g., the name *foot*).

The need for a universally standard and easy to remember system of units lead to the creation of the "System Internationale", or the *metric system*, which is used throughout the world (but not in the US!). In this system, the basic units are:

<i>distance:</i>	meter
<i>time:</i>	as English system
<i>mass:</i>	gram

All units are standardized: the meter is based on the wavelength of a certain color of light; the second is based on the vibration of a certain type of atom; the gram is based on the mass of one cubic centimeter (cc) of liquid water (why water?). In addition, the conversion between units is achieved by looking at the prefix:

<i>Giga:</i>	=	× 1 billion
<i>Mega:</i>	=	× 1 million (notice the "m's")
<i>kilo:</i>	=	× 1,000
<i>centi:</i>	=	× 1/100 (how many ¢ in a \$?)
<i>milli:</i>	=	× 1/1,000
<i>micro:</i>	=	× 1/1,000,000 ("m's" again)
<i>nano:</i>	=	× 1/1,000,000,000

Thus, a *kilometer* (km) is 1,000 meters (m), a *milligram* (mg) is *one one-thousandth* of a gram (g), and a *microsecond* (ms) is a *millionth* of a second (s). Because the units are all based on factors of ten, it is very easy to convert between them. Of course, some units are rarely used: centisecond and Megagram for example. Indeed, old habits can be hard to break: our class lasts 3.6 kiloseconds, but we usually just call it an hour!

Conversion between English and metric units is easy if you know the conversion factors. For instance:

$$\begin{array}{rcl} 1 \text{ in} & = & 2.5 \text{ cm} \\ 1 \text{ lb} & = & 0.45 \text{ kg} \end{array}$$

and so on. Try to think in the metric system as much as possible!

To further complicate matters, astronomers often invent their own units: the *Astronomical Unit*, the *light year*, and the *Angstrom* are a few examples. We will learn about these later.

Units can save effort and stress. Respect them, learn them, and use them!

3. Scientific Notation

Astronomy is almost unique among the sciences in the sizes of the numbers which must be dealt with. One moment we are worrying about the distances of electrons from the nucleus of an atom, in the next we might be pondering the scale of a galaxy. But instead of taking time writing all the zeros associated with these numbers (the Earth is about 150,000,000,000 meters, or 93,000,000 miles from the Sun), we have come up with a “shorthand” notation for expressing both very big and very small numbers. It is called “Scientific Notation” (read also page 16 of your text, where this is referred to as “Powers of Ten Notation”). The trick to it all is realizing that each of those “zeros” is equal to a factor of ten...

The first thing to learn is how to convert numbers back and forth between scientific notation and ordinary decimal notation. The expression “ 10^n ” where n is an integer (a whole number) simply means “10 raised to the n^{th} power,” or in other words, a number resulting from using 10 as a factor “ n ” times:

$$10^5 = 10 \times 10 \times 10 \times 10 \times 10 = 100,000$$

$$10^8 = 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 100,000,000$$

Notice that the number of zeros in the ordinary decimal expression is exactly equal to the power to which 10 is raised.

If the number is expressed in words, first write it down as an ordinary decimal number and then convert: thus, ten million equals 10,000,000; there are seven zeros, so in powers of ten notation, this is written as 10^7 .

A number which is some power of $1/10$ can also be expressed easily in scientific notation. By definition, $1/10 = 10^{-1}$ (the reason for this notation will become apparent later). More generally, the expression “ 10^{-n} ”, where n is an integer, means $1/10^n$. Thus,

$$10^{-3} = \frac{1}{10^3} = \frac{1}{1000}$$

and,

$$10^{-8} = \frac{1}{10^8} = \frac{1}{100000000}$$

But what about numbers that are not exactly powers of ten, such as 2000, 0.00003, etc.? Actually, they are only a little more complicated to write down than powers of ten. Take 2000 as an example. It is equal to:

$$2,000 = 2 \times 1,000$$

But $1,000 = 10^3$. So we can write:

$$2,000 = 2 \times 1,000 = 2 \times 10^3$$

As another example, what about 0.00003?

$$0.00003 = 3 \times \frac{1}{100000} = 3 \times 10^{-5}$$

because $1/100000 = 10^{-5}$.

Working with Scientific Notation

This last example may seem a little complicated, so here is a simple procedure for figuring this out. First write down the number as the number itself times 10^0 . For example, write 2000 as 2000×10^0 . We can do this because by definition $10^0 = 1$, and multiplying a number by 1 does not change it. What we now have has the general form:

$$\text{coefficient} \times 10^{\text{exponent}}$$

Right now (in this example), the exponent is 0, of course, but that will change shortly. In the above example, the coefficient is 2000.

Second, start moving the decimal point in the coefficient to the right or left. For each place you move it to the left, add 1 to the exponent. For each place that you move it to the right, subtract 1 from the exponent. What we are really doing is dividing (or multiplying) the coefficient by 10 each time, and at the same time multiplying or (dividing) the 10^{exponent} by 10 each time. So the overall value of the number does not change! For example:

$$0.00003 = 0.00003 \times 10^0 = 0.0003 \times 10^{-1} = 0.003 \times 10^{-2} = 0.03 \times 10^{-3} = 0.3 \times 10^{-4} = 3 \times 10^{-5}$$

Another example:

$$2000 = 2000 \times 10^0 = 200 \times 10^1 = 20 \times 10^2 = 2 \times 10^3$$

You should move the decimal point until there is exactly one non-zero digit to the left of the decimal point, as in the case of each example given. We then say that the number is expressed in standard form (a number like 0.003×10^{-2} is pretty silly). You should always express numbers in standard form. Notice also that you didn't have to write down each of the steps above; it is enough to just count the total number of places to move the decimal point and use that as the number to add or subtract from the exponent.

Examples:

$$250000 = 2.5 \times 10^5 \quad (\text{move 5 places to the left})$$

$$0.000035 = 3.5 \times 10^{-5} (\text{move 5 places to the right})$$

$$0.00000001 = 1 \times 10^{-8} \quad (\text{usually just written as } 10^{-8})$$

Math With Powers of Ten Notation

The rule for multiplying two numbers expressed in Scientific Notation is very easy: multiply the two coefficients to get the new coefficient, and add the exponents (algebraically - pay attention to minus signs!) to get the new exponent (remember earlier in the algebra section about how exponents add?). If either number has no coefficient, assume it is equal to 1. Thus, the following calculations can be done:

$$(4 \times 10^3) \times (2 \times 10^7) = 4 \times 2 \times 10^{3+7} = 8 \times 10^{10}$$

$$(2 \times 10^5) \times (2.5 \times 10^8) = 2 \times 2.5 \times 10^{-5+8} = 5.0 \times 10^3$$

$$(3 \times 10^{-7}) \times (3 \times 10^{-8}) = 3 \times 3 \times 10^{-7-8} = 9 \times 10^{-15}$$

$$(4 \times 10^7) \times (5 \times 10^5) = 4 \times 5 \times 10^{7+5} = 20 \times 10^{12} = 2.0 \times 10^{13} \quad (\text{note that this one had to go through an extra step at the end to get it into standard form.})$$

The rule for division is equally easy: divide the coefficients to get the new coefficient and subtract the exponent of the denominator for the exponent of the numerator to get the exponent of the result. Again, be careful about the signs of the exponents when subtracting; remember that subtracting a negative number is the same as adding a positive number. Thus, the following examples:

$$(6 \times 10^5) / (2 \times 10^3) = (6 / 2) \times 10^{5-3} = 3 \times 10^2$$

$$(9 \times 10^8) / (3 \times 10^{-5}) = (9/3) \times 10^{8-(-5)} = 3 \times 10^{13}$$

$$(5 \times 10^3) / (2 \times 10^7) = (5/2) \times 10^{3-7} = 2.5 \times 10^{-4}$$

If the result is not in standard form, then the decimal point of the coefficient should be moved to the left or right and the exponent should be increased or decreased as you have already learned to do, until the number is in standard form.